Mechanism of self-organization in double-species point vortex system

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Abstract

Mechanism of self-organization in unbounded, double-species, two-dimensional (2D) point vortex system is discussed. A kinetic equation is obtained using the Klimontovich formalism. No axisymmetric flow is assumed. The obtained collision term consists of a diffusion and a drift term similar to the Fokker-Planck type collision term. It is revealed that a mechanism for the 2D inverse cascade is mainly provided by the drift term, as the sign of the drift term for the negative vortices is opposite from the one for the positive vortices. When the system reaches a quasi-stationary state near the thermal equilibrium with negative absolute temperature, $d\omega/d\psi$ is expected to be positive, where ω is the vorticity and ψ the stream function. In this case, the diffusion term dissipate the mean field energy, while the drift term produces the mean field energy. As a whole, the mean field energy is conserved. Similarly, the diffusion term increases the entropy, while the drift term decreases the entropy. As a whole, the entropy production is positive (H theorem). It ensures that the system relaxes to the global thermal equilibrium state.

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I. INTRODUCTION

The two-dimensional (2D) point vortex system has been successfully applied to understand the various phenomena including 2D turbulence [1–3], neutral [4] and nonneutral [5, 6] plasmas. These phenomena share a common keyword, "inverse cascade". To tackle the problem, Onsager introduced a concept of negative temperature for the 2D point vortex system [7]. To understand the equilibrium state of the 2D point vortex system, or 2D flow, much research effort has been devoted. Joyce and Montgomery presented the sinh-Poisson equation which gives a thermal equilibrium distribution for the inviscid Euler equation [8]. However, surprisingly a time-asymptotic distribution traced by the decaying Navier-Stokes equation can be described by the sinh-Poisson equation [9]. Thus, another essential approach has been made to understand the relaxation process toward the thermal equilibrium state through the kinetic theory.

Collision term for the 2D single- and multi species point vortex system was presented, for example, by Chavanis, Dubin and O'Neil [5, 10–14]. However, these results have a problem that an axisymmetric, monotonically decreasing vorticity profile and an angular velocity profile do not relax towards the Boltzmann distribution.

In this paper, we discuss a mechanism of the self-organization in unbounded, double-species 2D point vortex system. We extend the kinetic theory of Yatsuyanagi [15] based on the Klimontovich formalism. A kinetic equation is obtained which has the Fokker-Planck type collision term consisting of a diffusion and a drift term. To derive an explicit formula of the collision term, the Klimontovich formalism is used. The obtained collision term has following good properties that resolve the above issues; (a) it conserves the mean field energy. (b) it satisfies the H theorem. (c) it vanishes when the system reaches a global thermal equilibrium state. The H theorem ensures that the system relaxes to the global thermal equilibrium state. It is also revealed that the drift term plays an important role in the 2D inverse cascade process. When the system reaches a quasi-stationary state near the thermal equilibrium with negative absolute temperature, $d\omega/d\psi$ is expected to be positive, where ω is the vorticity and ψ the stream function. In this case, the diffusion term dissipate the mean field energy, while the drift term produces the mean field energy. Similarly, the diffusion term increases the entropy, while the drift term decreases the entropy.

The organization of this paper is as follows. In Sec. II, the point vortex system is briefly

introduced. In Sec. III, we demonstrate explicit formulae for the diffusion and the drift terms as intermediate results. In Sec. IV, a detailed calculation of the diffusion term is shown. As the similar calculation can be applied to the drift term, details for the drift term are omitted. In Sec. V, three good properties of the collision term are demonstrated. Finally in Sec. VI, we discuss the important role of the drift term in the 2D inverse cascade with negative absolute temperature.

II. POINT VORTEX SYSTEM

Let us consider a 2D system consisting of N_+ positive and N_- negative point vortices. The circulation of each point vortex is given by either Ω or $-\Omega$ where Ω is a positive constant.

$$\hat{\omega} \equiv \hat{\omega}_+ + \hat{\omega}_-, \tag{1}$$

$$\hat{\omega}_+ = \sum_{i=1}^{N_+} \Omega\delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{2}$$

$$\hat{\omega}_- = -\sum_{i=N_+ +1}^{N_+ + N_-} \Omega\delta(\boldsymbol{r} - \boldsymbol{r}_i). \tag{3}$$

The position vector and the strength of the ith point vortex is given by $\boldsymbol{r}_i = \boldsymbol{r}_i(t)$ and Ω_i, respectively, $\hat{\omega} = \hat{\omega}(\boldsymbol{r}, t)$ is the z-component of the microscopic vorticity on the $x - y$ plane, and $\delta(\boldsymbol{r})$ is the Dirac delta function in two dimensions. For brevity, we shall omit the dependences on t and $\boldsymbol{r}$, if there is no ambiguity. The discretized vorticity (2) and (3) are formal solutions of the microscopic Euler equations. The microscopic variables in the microscopic equation are identified by $\hat{\cdot}$.

$$\frac{\partial}{\partial t}\hat{\omega}_+(\boldsymbol{r}, t) + \nabla \cdot (\hat{\boldsymbol{u}}(\boldsymbol{r}, t)\hat{\omega}_+(\boldsymbol{r}, t)) = 0, \tag{4}$$

$$\frac{\partial}{\partial t}\hat{\omega}_-(\boldsymbol{r}, t) + \nabla \cdot (\hat{\boldsymbol{u}}(\boldsymbol{r}, t)\hat{\omega}_-(\boldsymbol{r}, t)) = 0. \tag{5}$$

Note that the motion of the positive vortices is affected by the negative one through the velocity $\hat{\boldsymbol{u}}$ appearing in the convective term in Eq. (4), and vice versa. From now on, we shall denote the two equations for the positive and the negative vortices into the single formula with double-sign like:

$$\frac{\partial}{\partial t}\hat{\omega}_\pm(\boldsymbol{r}, t) + \nabla \cdot (\hat{\boldsymbol{u}}(\boldsymbol{r}, t)\hat{\omega}_\pm(\boldsymbol{r}, t)) = 0. \tag{6}$$

Other microscopic variables are defined by

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}(\boldsymbol{r}, t) = -\hat{\boldsymbol{z}} \times \nabla \hat{\psi}, \tag{7}$$

$$\begin{aligned} \hat{\psi} &= \hat{\psi}(\boldsymbol{r}, t) \\ &= \int d\boldsymbol{r}' G(\boldsymbol{r} - \boldsymbol{r}') \hat{\omega}(\boldsymbol{r}', t) \\ &= \sum_i \Omega_i G(\boldsymbol{r} - \boldsymbol{r}_i), \end{aligned} \tag{8}$$

$$G(\boldsymbol{r}) = -\frac{1}{2\pi} \ln |\boldsymbol{r}|, \tag{9}$$

where $\hat{\boldsymbol{u}}$ and $\hat{\psi}$ are the velocity field and the stream function in the 2D plane, $\hat{\boldsymbol{z}}$ is the unit vector in the z-direction, and $G(\boldsymbol{r})$ is the 2D Green function for the Laplacian operator with an infinite domain.

The microscopic Euler equations (6) are formally identical with the macroscopic Euler equation

$$\frac{\partial}{\partial t} \omega(\boldsymbol{r}, t) + \nabla \cdot (\boldsymbol{u}(\boldsymbol{r}, t) \omega(\boldsymbol{r}, t)) = 0 \tag{10}$$

where $\omega(\boldsymbol{r}, t) \equiv \langle \hat{\omega}(\boldsymbol{r}, t) \rangle$ and $\boldsymbol{u}(\boldsymbol{r}, t) \equiv \langle \hat{\boldsymbol{u}}(\boldsymbol{r}, t) \rangle$ are the macroscopic vorticity and the macroscopic velocity, respectively. Operator $\langle \cdot \rangle$ is an averaging operator. As a solution of a macroscopic fluid equation should be given by a smooth function, the singular solutions (2) and (3) should be regarded as not the solution of the macroscopic equation but the one of the microscopic equation. Thus we call the equation that has the microscopic point vortex solution, "microscopic" Euler equation.

Starting equation is the microscopic Euler equations (6). Following the Klimontovich formalism [16], the microscopic variables are decomposed into the macroscopic part and the fluctuation.

$$\hat{\omega}_{\pm} = \omega_{\pm} + \delta \omega_{\pm}, \tag{11}$$

$$\hat{\boldsymbol{u}} = \boldsymbol{u} + \delta \boldsymbol{u}, \tag{12}$$

Inserting Eqs. (11) and (12) into Eqs. (6), and taking the ensemble average, Eqs. (6) are rewritten as the following macroscopic equation with the diffusion fluxes $\boldsymbol{\Gamma}_{\pm} = \boldsymbol{\Gamma}_{\pm}(\boldsymbol{r}, t)$

$$\frac{\partial}{\partial t} \omega_{\pm} + \nabla \cdot (\boldsymbol{u} \omega_{\pm}) = -\nabla \cdot \boldsymbol{\Gamma}_{\pm}(\boldsymbol{r}, t), \tag{13}$$

$$\begin{aligned} \boldsymbol{\Gamma}_{\pm} &= \langle \delta \boldsymbol{u} \delta \omega_{\pm} \rangle \\ &= -\int d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}') \langle \delta \omega' \delta \omega_{\pm} \rangle, \end{aligned} \tag{14}$$

$$\boldsymbol{F}(\boldsymbol{r}) = \hat{\boldsymbol{z}} \times \nabla G(\boldsymbol{r}). \tag{15}$$

We note $\delta\omega'$ for $\delta\omega(\boldsymbol{r}', t)$. Similarly, we shall note ω' for $\omega(\boldsymbol{r}', t)$. The following relation is utilized.

$$\delta\boldsymbol{u} = -\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\delta\omega' \tag{16}$$

In the next section, we will analytically assess the diffusion fluxes $\boldsymbol{\Gamma}_\pm$.

III. EVALUATION OF DIFFUSION FLUX

To evaluate the diffusion fluxes $\boldsymbol{\Gamma}_\pm$ explicitly, we introduce a small parameter ϵ. We consider a point vortex system with large $N_\pm$ keeping the total circulation $N_\pm\Omega$ constant. The magnitude of ϵ is of the order of either $1/N_\pm$ or Ω. Orders of the other quantities are:

$$\begin{aligned}
&\boldsymbol{\Gamma}_\pm \approx O(\epsilon^2)\\
&\frac{\partial \boldsymbol{u}}{\partial t} \approx O(\epsilon), \quad \frac{\partial \omega}{\partial t} \approx O(\epsilon)\\
&\omega \approx \nabla^2\psi \approx O(\epsilon^0), \quad \nabla\omega \approx O(\epsilon),\\
&\boldsymbol{u} \approx \nabla\psi \approx O(\epsilon^0), \quad \nabla\boldsymbol{u} \approx \nabla^2\psi \approx O(\epsilon^0),\\
&\nabla\nabla\boldsymbol{u} \approx O(\epsilon).
\end{aligned}$$

The expansion parameter ϵ is the same one as Chavanis in Refs. [10, 17] and the references therein. Expressing $\boldsymbol{\Gamma}_\pm$ in the form of a perturbation expansion and gathering the second order terms, analytical formulae for the diffusion fluxes $\boldsymbol{\Gamma}_\pm$ will be obtained.

To rewrite the diffusion fluxes in Eq. (13) according to the above prospect, we introduce linearized equations obtained by inserting Eqs. (11) and (12) into Eqs. (6) and assembling the first-order fluctuation terms:

$$\frac{\partial}{\partial t}\delta\omega_\pm + \nabla\cdot(\boldsymbol{u}\delta\omega_\pm) = -\delta\boldsymbol{u}\cdot\nabla\omega_\pm. \tag{17}$$

As the macroscopic quantities $\boldsymbol{u}$ appearing in the second term in the left-hand side and $\nabla\omega$ in the right-hand side are supposed to be constant in the time scale of the microscopic fluctuation, Eqs. (17) can be integrated:

$$\begin{aligned}
\delta\omega_\pm = &-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r}-\boldsymbol{u}(t-\tau),\tau)\cdot\nabla\omega_\pm\\
&+\delta\omega_\pm(\boldsymbol{r}-\boldsymbol{u}(t-t_0),t_0),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\delta\omega'_\pm = &-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r}'-\boldsymbol{u}'(t-\tau),\tau)\cdot\nabla'\omega'_\pm\\
&+\delta\omega_\pm(\boldsymbol{r}'-\boldsymbol{u}'(t-t_0),t_0),
\end{aligned} \tag{19}$$

where $\nabla'\omega'_\pm = \nabla_{\boldsymbol{r}'}\omega_\pm(\boldsymbol{r}')$. We call this approximation "straight-line approximation" where the trajectory of the point vortex is straight [12]. The value of t_0 is chosen to satisfy $t - t_0 \gg t_c$ where t_c is a correlation time of the fluctuation. Substituting Eqs. (18) and (19) into the correlation terms in Eqs. (14), we obtain

$$
\begin{aligned}
&\langle\delta\omega'\delta\omega_\pm\rangle \\
&= \left\langle\left(-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r}' - \boldsymbol{u}'(t-\tau),\tau)\cdot\nabla'\omega'\right)\right. \\
&\quad\left.\times\left(-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r} - \boldsymbol{u}(t-\tau),\tau)\cdot\nabla\omega_\pm\right)\right\rangle \\
&\quad+\left\langle\left(-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r}' - \boldsymbol{u}'(t-\tau),\tau)\cdot\nabla'\omega'\right)\delta\omega_\pm(\boldsymbol{r} - \boldsymbol{u}(t-t_0),t_0)\right\rangle \\
&\quad+\left\langle\delta\omega(\boldsymbol{r}' - \boldsymbol{u}'(t-t_0),t_0)\left(-\int_{t_0}^{t} d\tau\delta\boldsymbol{u}(\boldsymbol{r} - \boldsymbol{u}(t-\tau),\tau)\cdot\nabla\omega_\pm\right)\right\rangle \\
&\quad+\langle\delta\omega(\boldsymbol{r}' - \boldsymbol{u}'(t-t_0),t_0)\delta\omega_\pm(\boldsymbol{r} - \boldsymbol{u}(t-t_0),t_0)\rangle \qquad (20)\\
&= -\int_{t_0}^{t} d\tau\,\langle\delta\boldsymbol{u}(\boldsymbol{r} - \boldsymbol{u}(t-\tau),\tau)\delta\omega'\rangle\cdot\nabla\omega_\pm \\
&\quad-\int_{t_0}^{t} d\tau\,\langle\delta\boldsymbol{u}(\boldsymbol{r}' - \boldsymbol{u}'(t-\tau),\tau)\delta\omega_\pm\rangle\cdot\nabla'\omega' + O(\epsilon^3) \qquad (21)\\
&\approx \int_{t_0}^{t} d\tau\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r} - \boldsymbol{u}(t-\tau) - \boldsymbol{r}'')\cdot\nabla\omega_\pm\,\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega'\rangle \\
&\quad+\int_{t_0}^{t} d\tau\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r}' - \boldsymbol{u}'(t-\tau) - \boldsymbol{r}'')\cdot\nabla'\omega'\,\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega_\pm\rangle\,. \qquad (22)
\end{aligned}
$$

When obtaining formulae (21), we assume that the first terms in formulae (20) are negligible as they have two nablas. We drop the last terms also as they should have a factor of $1/(t-t_0)$ and we focus on $t - t_0 \gg t_c$ case. The time is shifted from t_0 to t using the straight-line approximation. When rewriting formulae (21) as (22), Eq. (16) is used.

Inserting Eq. (22) into Eqs. (14), the following intermediate results are obtained:

$$
\begin{aligned}
&\boldsymbol{\Gamma}_\pm \\
&= -\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r} - \boldsymbol{u}(t-\tau) - \boldsymbol{r}'')\cdot\nabla\omega_\pm \\
&\quad\times\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega'\rangle \\
&\quad-\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}' - \boldsymbol{u}'(t-\tau) - \boldsymbol{r}'')\cdot\nabla'\omega' \\
&\quad\times\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega_\pm\rangle\,. \qquad (23)
\end{aligned}
$$

IV. EVALUATION OF THE FIRST TERM OF THE DIFFUSION FLUX

There are three correlation terms $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega' \rangle$, $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega_+ \rangle$ and $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega_- \rangle$ in Eqs. (23). As the expression of the correlation terms $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega_+ \rangle$ and $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega_- \rangle$ are very similar to $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega' \rangle$, only the detailed derivation for $\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega' \rangle$ will be shown:

$$
\begin{aligned}
&\langle \delta\omega(\boldsymbol{r}'', \tau)\delta\omega' \rangle \\
&= \langle (\hat{\omega}(\boldsymbol{r}'', \tau) - \omega(\boldsymbol{r}'', \tau)) (\hat{\omega}' - \omega') \rangle \\
&= \langle (\hat{\omega}_+(\boldsymbol{r}'', \tau) + \hat{\omega}_-(\boldsymbol{r}'', \tau) - \omega_+(\boldsymbol{r}'', \tau) - \omega_-(\boldsymbol{r}'', \tau)) \\
&\quad \times \left(\hat{\omega}'_+ + \hat{\omega}'_- - \omega'_+ - \omega'_- \right) \rangle \\
&= \left\langle (\hat{\omega}_+(\boldsymbol{r}'', \tau) + \hat{\omega}_-(\boldsymbol{r}'', \tau)) \times \left(\hat{\omega}'_+ + \hat{\omega}'_- \right) \right\rangle \\
&\quad - (\omega_+(\boldsymbol{r}'', \tau) + \omega_-(\boldsymbol{r}'', \tau)) \times \left(\omega'_+ + \omega'_- \right) \\
&= \left\langle \sum_{i=1}^{N_+ + N_-} \Omega_i^2 \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle \\
&\quad + \left\langle \sum_{i=1}^{N_+ + N_-} \sum_{j \neq i}^{N_+ + N_-} \Omega_i \Omega_j \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \\
&\quad - \omega(\boldsymbol{r}'', \tau)\omega'.
\end{aligned} \tag{24}
$$

The first term in the last result in Eq. (24) corresponds to the case of $i = j$, and the second term corresponds to the case of $i \neq j$.

For the $i = j$ case, the formula is rewritten as

$$
\begin{aligned}
&\left\langle \sum_{i=1}^{N_+ + N_-} \Omega_i^2 \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle \\
&= \left\langle \sum_{i=1}^{N_+ + N_-} \Omega_i^2 \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau) - \boldsymbol{r}' + \boldsymbol{r}_i) \delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle \\
&= \sum_{i=1}^{N_+ + N_-} \Omega_i^2 \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau) - \boldsymbol{r}' + \boldsymbol{r}_i) \delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle .
\end{aligned} \tag{25}
$$

Here we introduce a stochastic process to evaluate $\boldsymbol{r}_i - \boldsymbol{r}_i(\tau)$:

$$
\begin{aligned}
\boldsymbol{r}_i - \boldsymbol{r}_i(\tau) &= \int_\tau^t \boldsymbol{u}(\boldsymbol{r}(\tau'), \tau') d\tau' + \boldsymbol{\xi} \\
&\approx \boldsymbol{u}'(t - \tau) + \boldsymbol{\xi}.
\end{aligned} \tag{26}
$$

The first term in Eq. (26) represents the straight-line approximation and the second term represents a Brownian motion. The stochastic process represented by $\langle \cdot \rangle_\xi$ includes all the

possible motion to reach position $\boldsymbol{r}_i$ at time t. Then, Eq. (25) can be rewritten as

$$
\begin{aligned}
&\sum_{i=1}^{N_+ + N_-} \Omega_i^2 \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau) - \boldsymbol{r}' + \boldsymbol{r}_i)\delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle \\
&= \sum_{i=1}^{N_+ + N_-} \Omega_i^2 \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}' + \boldsymbol{u}'(t-\tau) + \boldsymbol{\xi}) \right\rangle \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_i) \right\rangle \\
&= \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}' + \boldsymbol{u}'(t-\tau) + \boldsymbol{\xi}) \right\rangle_\xi \Omega(\omega'_+ - \omega'_-)
\end{aligned}
\tag{27}
$$

For the $i \neq j$ case, we introduce an approximation that the correlation between the particles can be neglected

$$
\begin{aligned}
&\sum_{i}^{N_+ + N_-} \sum_{i \neq j}^{N_+ + N_-} \Omega_i \Omega_j \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau))\delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \\
&\approx \sum_{i}^{N_+ + N_-} \sum_{j \neq i}^{N_+ + N_-} \Omega_i \Omega_j \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle .
\end{aligned}
\tag{28}
$$

Also we assume the followings:

$$
\sum_{i}^{N_+ + N_-} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle = \omega_+(\boldsymbol{r}'', \tau) + \omega_-(\boldsymbol{r}'', \tau) \tag{29}
$$

$$
\omega_+(\boldsymbol{r}'', \tau) = N_+ \Omega \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle , \tag{30}
$$

$$
\omega_-(\boldsymbol{r}'', \tau) = -N_- \Omega \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle . \tag{31}
$$

Inserting Eqs. (30) and (31) into Eq. (28), we obtain

$$
\begin{aligned}
&\sum_{i}^{N_+ + N_-} \sum_{j \neq i}^{N_+ + N_-} \Omega_i \Omega_j \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \\
&= \left(\sum_{i=1}^{N_+ + N_-} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \right) \times \left(\sum_{j \neq i}^{N_+ + N_-} \Omega_j \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \right) \\
&= \left(\sum_{i=1}^{N_+} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \right) \times \left(\sum_{j=1 \neq i}^{N_+} \Omega_j \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \right) \\
&\quad + \left(\sum_{i=1}^{N_+} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \right) \times \left(\sum_{j=N_+ + 1}^{N_+ + N_-} \Omega_j \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \right) \\
&\quad + \left(\sum_{i=N_+ + 1}^{N_+ + N_-} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \right) \times \left(\sum_{j=1}^{N_+} \Omega_j \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \right) \\
&\quad + \left(\sum_{i=N_+ + 1}^{N_+ + N_-} \Omega_i \left\langle \delta(\boldsymbol{r}'' - \boldsymbol{r}_i(\tau)) \right\rangle \right) \times \left(\sum_{j=N_+ + 1 \neq i}^{N_+ + N_-} \Omega_j \left\langle \delta(\boldsymbol{r}' - \boldsymbol{r}_j) \right\rangle \right) \\
&= N_+ \Omega \frac{\omega_+(\boldsymbol{r}'', \tau)}{N_+ \Omega} (N_+ - 1) \Omega \frac{\omega'_+}{N_+ \Omega}
\end{aligned}
$$

$$
\begin{aligned}
&+N_+\Omega\frac{\omega_+(\boldsymbol{r}'',\tau)}{N_+\Omega}N_-\Omega\frac{\omega'_-}{N_-\Omega}\\
&+N_-\Omega\frac{\omega_-(\boldsymbol{r}'',\tau)}{N_-\Omega}N_+\Omega\frac{\omega'_+}{N_+\Omega}\\
&+N_-\Omega\frac{\omega_-(\boldsymbol{r}'',\tau)}{N_-\Omega}(N_--1)\Omega\frac{\omega'_-}{N_-\Omega}\\
=&\ (\omega_+(\boldsymbol{r}'',\tau)+\omega_-(\boldsymbol{r}'',\tau))\times\left(\omega'_++\omega'_-\right)\\
&-\frac{1}{N_+}\omega_+(\boldsymbol{r}'',\tau)\omega'_+-\frac{1}{N_-}\omega_-(\boldsymbol{r}'',\tau)\omega'_-.
\end{aligned}
\tag{32}
$$

Combining the results of the $i=j$ and the $i\neq j$ cases, we rewrite Eq. (24) as

$$
\begin{aligned}
&\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega'\rangle\\
=&\ \Omega\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}'+\boldsymbol{u}'(t-\tau)+\boldsymbol{\xi})\right\rangle_\xi\left(\omega'_+-\omega'_-\right)\\
&-\frac{1}{N_+}\omega_+(\boldsymbol{r}'',\tau)\omega'_+-\frac{1}{N_-}\omega_-(\boldsymbol{r}'',\tau)\omega'_-).
\end{aligned}
\tag{33}
$$

Similarly, we obtain

$$
\begin{aligned}
&\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega_\pm\rangle\\
=&\ \pm\Omega\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}+\boldsymbol{u}(t-\tau)+\boldsymbol{\xi})\right\rangle_\xi\omega_\pm\\
&-\frac{1}{N_\pm}\omega_\pm(\boldsymbol{r}'',\tau)\omega_\pm.
\end{aligned}
\tag{34}
$$

At first, we focus on the evaluation of Eq. (33). After that, we will return to the evaluation of Eqs. (34). As we request the total circulation $N_\pm\Omega$ constant, the three terms in the right hand side of Eq. (33) are of the same order. To proceed with the evaluation of Eq. (33), a conservation law is introduced

$$
\int d\boldsymbol{r}'\left\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega'\right\rangle=0.
\tag{35}
$$

Inserting Eq. (33) into Eq. (35), we obtain

$$
\begin{aligned}
&\int d\boldsymbol{r}'\Omega\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}'+\boldsymbol{u}'(t-\tau)+\boldsymbol{\xi})\right\rangle_\xi(\omega'_+-\omega'_-)\\
&-\frac{1}{N_+}\omega_+(\boldsymbol{r}'',\tau)\omega'_+-\frac{1}{N_-}\omega_-(\boldsymbol{r}'',\tau)\omega'_-\\
=&\ \Omega\int d\boldsymbol{r}'\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}'+\boldsymbol{u}'(t-\tau)+\boldsymbol{\xi})\right\rangle_\xi\left(\omega'_+-\omega'_-\right)\\
&-\frac{1}{N_+}\omega_+(\boldsymbol{r}'',\tau)\int\omega'_+d\boldsymbol{r}'-\frac{1}{N_-}\omega_-(\boldsymbol{r}'',\tau)\int\omega'_-d\boldsymbol{r}'\\
=&\ \Omega\int d\boldsymbol{r}'\left\langle\delta(\boldsymbol{r}''-\boldsymbol{r}'+\boldsymbol{u}'(t-\tau)+\boldsymbol{\xi})\right\rangle_\xi\left(\omega'_+-\omega'_-\right)\\
&-\frac{1}{N_+}\omega_+(\boldsymbol{r}'',\tau)N_+\Omega-\frac{1}{N_-}\omega_-(\boldsymbol{r}'',\tau)N_-\Omega\\
=&\ 0.
\end{aligned}
\tag{36}
$$

This equation yields

$$
\begin{aligned}
&\omega_\pm(\boldsymbol{r}'',\tau)\\
&= \pm\int d\boldsymbol{q}'\,\langle\delta(\boldsymbol{r}''-\boldsymbol{q}'+\boldsymbol{u}(\boldsymbol{q}')(t-\tau)+\boldsymbol{\xi})\rangle_\xi\,\omega_\pm(\boldsymbol{q}',t)
\end{aligned}
\tag{37}
$$

where $d\boldsymbol{r}'$ is replaced by $d\boldsymbol{q}'$ to avoid ambiguity. This equation enables that all the quantities at τ is converted by the ones at t. Inserting Eqs. (33) and (37) into the first term in Eq. (23), we obtain

$$
\begin{aligned}
&-\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{u}(t-\tau)-\boldsymbol{r}'')\cdot\nabla\omega_\pm\\
&\times\langle\delta\omega(\boldsymbol{r}'',\tau)\delta\omega'\rangle\\
=&-\Omega\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{u}(t-\tau)-\boldsymbol{r}'')\cdot\nabla\omega_\pm\\
&\times\langle\delta(\boldsymbol{r}''-\boldsymbol{r}'+\boldsymbol{u}'(t-\tau)+\boldsymbol{\xi})\rangle_\xi\left(\omega_+'-\omega_-'\right)\\
&+\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{u}(t-\tau)-\boldsymbol{r}'')\cdot\nabla\omega_\pm\\
&\quad\times\left[\frac{1}{N_+}\omega_+'\int d\boldsymbol{q}'\,\langle\delta(\boldsymbol{r}''-\boldsymbol{q}'+\boldsymbol{u}(\boldsymbol{q}')(t-\tau)+\boldsymbol{\xi})\rangle_\xi\,\omega_+(\boldsymbol{q}',t)\right.\\
&\quad\left.-\frac{1}{N_-}\omega_-'\int d\boldsymbol{q}'\,\langle\delta(\boldsymbol{r}''-\boldsymbol{q}'+\boldsymbol{u}(\boldsymbol{q}')(t-\tau)+\boldsymbol{\xi})\rangle_\xi\,\omega_-(\boldsymbol{q}',t)\right].
\end{aligned}
\tag{38}
$$

We proceed with the evaluation of the second term in Eqs. (38):

$$
\begin{aligned}
&\int_{t_0}^{t} d\tau\int d\boldsymbol{r}'\int d\boldsymbol{r}''\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{u}(t-\tau)-\boldsymbol{r}'')\cdot\nabla\omega_\pm\\
&\quad\times\left[\frac{1}{N_+}\omega_+'\int d\boldsymbol{q}'\,\langle\delta(\boldsymbol{r}''-\boldsymbol{q}'+\boldsymbol{u}(\boldsymbol{q}')(t-\tau)+\boldsymbol{\xi})\rangle_\xi\,\omega_+(\boldsymbol{q}',t)\right.\\
&\quad\left.-\frac{1}{N_-}\omega_-'\int d\boldsymbol{q}'\,\langle\delta(\boldsymbol{r}''-\boldsymbol{q}'+\boldsymbol{u}(\boldsymbol{q}')(t-\tau)+\boldsymbol{\xi})\rangle_\xi\,\omega_-(\boldsymbol{q}',t)\right]\\
=&\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\int_{t_0}^{t} d\tau\int d\boldsymbol{q}'\,\langle\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{q}'-(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))(t-\tau)+\boldsymbol{\xi})\rangle_\xi\cdot\nabla\omega_\pm\\
&\quad\times\left[\frac{1}{N_+}\omega_+(\boldsymbol{q}',t)\omega_+'-\frac{1}{N_-}\omega_-(\boldsymbol{q}',t)\omega_-'\right]
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
=&\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\int_{t_0}^{t} d\tau\int\frac{d\boldsymbol{k}}{(2\pi)^2}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\nabla\omega_\pm\\
&\times\int d\boldsymbol{q}'\exp\left[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{q}'-(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))(t-\tau))\right]\langle\exp(i\boldsymbol{k}\cdot\boldsymbol{\xi})\rangle_\xi\\
&\times\left(\frac{1}{N_+}\omega_+(\boldsymbol{q}',t)\omega_+'-\frac{1}{N_-}\omega_-(\boldsymbol{q}',t)\omega_-'\right).
\end{aligned}
\tag{40}
$$

To rewrite formulae (39) as (40), we use the Fourier transformation:

$$
\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{q}'-(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))(t-\tau))
$$

$$= \frac{1}{(2\pi)^2}\int d\boldsymbol{k}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\exp(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{q}'-(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))(t-\tau))). \tag{41}$$

The term $\langle \exp(i\boldsymbol{k}\cdot\boldsymbol{\xi})\rangle_\xi$ represents a Brownian motion of the point vortices with diffusion tensor D and is evaluated by the cumulant expansion:

$$\begin{aligned}\langle \exp(i\boldsymbol{k}\cdot\boldsymbol{\xi})\rangle_\xi &= \exp\left(-\frac{\boldsymbol{k}\cdot\mathsf{D}\cdot\boldsymbol{k}}{2}(t-\tau)\right)\\ &\equiv \exp(-\nu(t-\tau))\end{aligned} \tag{42}$$

where ν is a small positive parameter. Inserting the following formula into Eqs. (40),

$$\begin{aligned}&\int_{t_0}^{t} d\tau \exp[-i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}')-\nu)(t-\tau)]\\ &\approx \pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))|^2+\nu^2},\end{aligned} \tag{43}$$

we obtain

$$\begin{aligned}&\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\int_{t_0}^{t}d\tau\int\frac{d\boldsymbol{k}}{(2\pi)^2}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\nabla\omega_\pm\\ &\times\int d\boldsymbol{q}'\exp\left[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{q}'-(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))(t-\tau))\right]\langle\exp(i\boldsymbol{k}\cdot\boldsymbol{\xi})\rangle_\xi\\ &\times\left[\frac{1}{N_+}\omega_+(\boldsymbol{q}',t)\omega'_+ - \frac{1}{N_-}\omega_-(\boldsymbol{q}',t)\omega'_-\right]\\ =&\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\int\frac{d\boldsymbol{k}}{(2\pi)^2}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\nabla\omega_\pm\\ &\times\int d\boldsymbol{q}'\left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))-\frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))|^2+\nu^2}\right]\exp(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{q}'))\\ &\times\left[\frac{1}{N_+}\omega_+(\boldsymbol{q}',t)\omega'_+ - \frac{1}{N_-}\omega_-(\boldsymbol{q}',t)\omega'_-\right].\end{aligned} \tag{44}$$

We substitute $\boldsymbol{r}+\boldsymbol{q}''$ for $\boldsymbol{q}'$ and expand $\boldsymbol{u}(\boldsymbol{q}')$ and $\omega(\boldsymbol{q}')$ in the form of the Taylor series and retain the zero-th order terms only:

$$\boldsymbol{u}(\boldsymbol{q}') = \boldsymbol{u}(\boldsymbol{r}) + \boldsymbol{q}''\cdot\nabla\boldsymbol{u}(\boldsymbol{r}) + O(\epsilon), \tag{45}$$

$$\omega(\boldsymbol{q}') = \omega(\boldsymbol{r}+\boldsymbol{q}'') = \omega(\boldsymbol{r}) + O(\epsilon). \tag{46}$$

Inserting Eqs. (45) and (46) into Eqs. (44), we finally obtain

$$\begin{aligned}&\int d\boldsymbol{r}'\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\int\frac{d\boldsymbol{k}}{(2\pi)^2}\frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2}\cdot\nabla\omega_\pm\\ &\times\int d\boldsymbol{q}'\left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))-\frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}(\boldsymbol{q}'))|^2+\nu^2}\right]\exp(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{q}'))\\ &\times\left[\frac{1}{N_+}\omega_+(\boldsymbol{q}',t)\omega'_+ - \frac{1}{N_-}\omega_-(\boldsymbol{q}',t)\omega'_-\right]\end{aligned}$$

$$
\begin{aligned}
&= \int d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}') \int \frac{d\boldsymbol{k}}{(2\pi)^2} \frac{\hat{\boldsymbol{z}} \times i\boldsymbol{k}}{|\boldsymbol{k}|^2} \cdot \nabla\omega_\pm \\
&\quad \times \int d\boldsymbol{q}'' \left[\pi\delta(-\boldsymbol{k}\cdot(\boldsymbol{q}''\cdot\nabla)\boldsymbol{u}) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{q}''\cdot\nabla)\boldsymbol{u}}{|\boldsymbol{k}\cdot(\boldsymbol{q}''\cdot\nabla)\boldsymbol{u}|^2+\nu^2} \right] \exp(-i\boldsymbol{k}\cdot\boldsymbol{q}'') \\
&\quad\quad \times \left(\frac{1}{N_+}\omega_+\omega'_+ - \frac{1}{N_-}\omega_-\omega'_- \right).
\end{aligned} \tag{47}
$$

It is found that by substituting $\boldsymbol{k} = -\boldsymbol{k}$ and $\boldsymbol{q}'' = -\boldsymbol{q}''$, the sign of Eqs. (47) changes. Thus it is concluded that the integral equals zero, i.e. the second term in Eq. (38) has zero contribution and only the first term remains. The obtained formulae for the first terms of the diffusion fluxes (23) are as follows:

$$
\begin{aligned}
&- \int_{t_0}^{t} d\tau \int d\boldsymbol{r}' \int d\boldsymbol{r}'' \boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}-\boldsymbol{u}(t-\tau)-\boldsymbol{r}'') \cdot \nabla\omega_\pm \\
&\quad \times \langle \delta\omega(\boldsymbol{r}'',\tau)\delta\omega' \rangle \\
&= -\Omega \int d\boldsymbol{r}' \boldsymbol{F}(\boldsymbol{r}-\boldsymbol{r}') \int \frac{d\boldsymbol{k}}{(2\pi)^2} \exp(i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')) \frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2} \cdot \left(\omega'_+ - \omega'_-\right) \nabla\omega_\pm \\
&\quad \times \left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2+\nu^2} \right].
\end{aligned} \tag{48}
$$

The similar calculations can be adapted for the remaining two terms with $\langle \delta\omega(\boldsymbol{r}'',\tau)\delta\omega_+ \rangle$ and $\langle \delta\omega(\boldsymbol{r}'',\tau)\delta\omega_- \rangle$. For these cases, the following conservation laws are used:

$$
\int d\boldsymbol{r} \, \langle \delta\omega(\boldsymbol{r}'',\tau)\delta\omega_\pm(\boldsymbol{r},t) \rangle = 0. \tag{49}
$$

The whole results including both the diffusion and the drift terms are given by

$$
\begin{aligned}
\boldsymbol{\Gamma}_\pm = -\Omega \int d\boldsymbol{r}' \int \frac{d\boldsymbol{k}}{(2\pi)^2} \int \frac{d\boldsymbol{k}'}{(2\pi)^2} \exp(i(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{r}')) \\
\times \left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2+\nu^2} \right] \\
\times \frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}'}{|\boldsymbol{k}'|^2} \frac{\hat{\boldsymbol{z}}\times i\boldsymbol{k}}{|\boldsymbol{k}|^2} \cdot \left[(\omega'_+ - \omega'_-)\nabla\omega_\pm \mp \omega_\pm \nabla'\omega' \right]
\end{aligned} \tag{50}
$$

where we have used Eq. (41). It should be noted that the obtained diffusion fluxes can be divided into two parts, namely the diffusion tensor $\mathsf{D}(\boldsymbol{r},t)$ proportional to $\nabla\omega_\pm$ and the drift velocity $\boldsymbol{V}(\boldsymbol{r},t)$ proportional to $\omega_\pm$.

$$
\begin{aligned}
\boldsymbol{\Gamma}_\pm &= -\mathsf{D}\cdot\nabla\omega_\pm \pm \boldsymbol{V}\omega_\pm, \qquad (51) \\
\mathsf{D} &= \Omega \int d\boldsymbol{r}' \int \frac{d\boldsymbol{k}}{(2\pi)^2} \int \frac{d\boldsymbol{k}'}{(2\pi)^2} \exp(i(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{r}')) \\
&\quad \times \left[\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')) - \frac{i\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2+\nu^2} \right]
\end{aligned}
$$

$$\times \frac{\hat{\boldsymbol{z}} \times i\boldsymbol{k}'}{|\boldsymbol{k}'|^2} \frac{\hat{\boldsymbol{z}} \times i\boldsymbol{k}}{|\boldsymbol{k}|^2} (\omega'_+ - \omega'_-) \tag{52}$$

$$\begin{aligned}
\boldsymbol{V} = \Omega \int d\boldsymbol{r}' \int \frac{d\boldsymbol{k}}{(2\pi)^2} \int \frac{d\boldsymbol{k}'}{(2\pi)^2} \exp(i(\boldsymbol{k}+\boldsymbol{k}') \cdot (\boldsymbol{r} - \boldsymbol{r}')) \\
\times \left[\pi\delta(\boldsymbol{k} \cdot (\boldsymbol{u} - \boldsymbol{u}')) - \frac{i\boldsymbol{k} \cdot (\boldsymbol{u} - \boldsymbol{u}')}{|\boldsymbol{k} \cdot (\boldsymbol{u} - \boldsymbol{u}')|^2 + \nu^2} \right] \\
\times \frac{\hat{\boldsymbol{z}} \times i\boldsymbol{k}'}{|\boldsymbol{k}'|^2} \frac{\hat{\boldsymbol{z}} \times i\boldsymbol{k}}{|\boldsymbol{k}|^2} \cdot \nabla'\omega'
\end{aligned} \tag{53}$$

V. SPACE-AVERAGED COLLISION TERM

Equation (50) includes the oscillatory term $\exp(i(\boldsymbol{k}+\boldsymbol{k}') \cdot (\boldsymbol{r}-\boldsymbol{r}'))$. To reveal the characteristics of the obtained collision term, we need to calculate the space average of the diffusion fluxes to drop the high-frequency component. Space average is calculated over the small rectangular area $\Lambda(\boldsymbol{r})$ with sides both $2L$ located at $\boldsymbol{r}$. The space average of the diffusion flux $\boldsymbol{\Gamma}_\pm$ is defined by

$$\langle \boldsymbol{\Gamma}_\pm \rangle_s \equiv \boldsymbol{\Gamma}_{s\pm}(\boldsymbol{r}) = \frac{1}{|\Lambda(\boldsymbol{r})|} \int_{\Lambda(\boldsymbol{r})} d\boldsymbol{r}'' \boldsymbol{\Gamma}_\pm(\boldsymbol{r}''). \tag{54}$$

We assume that the macroscopic variables such as $\boldsymbol{u}$ and ω may be constant inside $\Lambda(\boldsymbol{r})$ and only the term $\exp(i(\boldsymbol{k}+\boldsymbol{k}') \cdot (\boldsymbol{r}-\boldsymbol{r}'))$ should be space-averaged:

$$\begin{aligned}
&\langle \exp(i(\boldsymbol{k}+\boldsymbol{k}') \cdot (\boldsymbol{r}-\boldsymbol{r}')) \rangle_s \\
&= \frac{1}{(2L)^2} \int_{-L}^{L} dx'' \int_{-L}^{L} dy'' \exp(i(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}'') \exp(-i(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}') \\
&\approx \left(\frac{\pi}{L}\right)^2 \delta(\boldsymbol{k}+\boldsymbol{k}') \exp(-i(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}') \\
&= \left(\frac{\pi}{L}\right)^2 \delta(\boldsymbol{k}+\boldsymbol{k}')
\end{aligned} \tag{55}$$

where $\boldsymbol{r}'' = (x'', y'')$. Thus, space-averaged diffusion fluxes are given by

$$\begin{aligned}
&\boldsymbol{\Gamma}_{s\pm}(\boldsymbol{r}) \\
&= -\Omega \left(\frac{\pi}{L}\right)^2 \int d\boldsymbol{r}' \int \frac{d\boldsymbol{k}}{(2\pi)^4} \pi\delta(\boldsymbol{k} \cdot (\boldsymbol{u} - \boldsymbol{u}')) \\
&\quad \times \frac{\hat{\boldsymbol{z}} \times \boldsymbol{k}}{|\boldsymbol{k}|^2} \frac{\hat{\boldsymbol{z}} \times \boldsymbol{k}}{|\boldsymbol{k}|^2} \cdot \left[(\omega'_+ - \omega'_-)\nabla\omega \mp \omega_\pm \nabla'\omega' \right].
\end{aligned} \tag{56}$$

In Eqs. (56), we omit the imaginary part as the collision term consists of only the real part. Further integration over $\boldsymbol{k}$ in Eqs. (56) can be performed. The integral concerning $\boldsymbol{k}$ is as

follows:

$$\int d\boldsymbol{k}\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))\frac{(\hat{\boldsymbol{z}}\times\boldsymbol{k})(\hat{\boldsymbol{z}}\times\boldsymbol{k})}{|\boldsymbol{k}^4|}. \tag{57}$$

Dividing $\boldsymbol{k}$ into the parallel and the perpendicular components and inserting them into Eq. (57),

$$\begin{aligned}
\boldsymbol{k} &= k_\parallel \hat{\boldsymbol{n}}_\parallel + k_\perp \hat{\boldsymbol{n}}_\perp,\\
\hat{\boldsymbol{n}}_\parallel &= \frac{\boldsymbol{u}-\boldsymbol{u}'}{|\boldsymbol{u}-\boldsymbol{u}'|},\\
\hat{\boldsymbol{n}}_\perp &= \hat{\boldsymbol{z}}\times\hat{\boldsymbol{n}}_\parallel,
\end{aligned} \tag{58}$$

we obtain

$$\begin{aligned}
&\int d\boldsymbol{k}\delta(\boldsymbol{k}\cdot(\boldsymbol{u}-\boldsymbol{u}'))\frac{(\hat{\boldsymbol{z}}\times\boldsymbol{k})(\hat{\boldsymbol{z}}\times\boldsymbol{k})}{|\boldsymbol{k}^4|}\\
&= \int dk_\parallel \int dk_\perp \delta(k_\parallel|\boldsymbol{u}-\boldsymbol{u}'|)\\
&\qquad \times\frac{[\hat{\boldsymbol{z}}\times(k_\parallel\hat{\boldsymbol{n}}_\parallel+k_\perp\hat{\boldsymbol{n}}_\perp)][\hat{\boldsymbol{z}}\times(k_\parallel\hat{\boldsymbol{n}}_\parallel+k_\perp\hat{\boldsymbol{n}}_\perp)]}{|k_\parallel^2+k_\perp^2|^2}\\
&= \int dk_\perp \frac{1}{|\boldsymbol{u}-\boldsymbol{u}'|}\frac{1}{k_\perp^4}(\hat{\boldsymbol{z}}\times k_\perp\hat{\boldsymbol{n}}_\perp)(\hat{\boldsymbol{z}}\times k_\perp\hat{\boldsymbol{n}}_\perp)\\
&= \int dk_\perp \frac{1}{|\boldsymbol{u}=\boldsymbol{u}'|}\frac{1}{k_\perp^2}\frac{\boldsymbol{u}-\boldsymbol{u}'}{|\boldsymbol{u}-\boldsymbol{u}'|}\frac{1}{k_\perp^2}\frac{\boldsymbol{u}-\boldsymbol{u}'}{|\boldsymbol{u}-\boldsymbol{u}'|}\\
&= \frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}2\left[-\frac{1}{k_\perp}\right]_{k_{\min}}^{\infty}\\
&= \frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\frac{2}{k_{\min}}
\end{aligned} \tag{59}$$

where parameter $k_{\min}$ is introduced to regularize a singularity and determined by the largest wave length that does not exceed a system size, namely $k_{\min} = 2\pi/R$ where R is a characteristic system size determined by an initial distribution of the vortices.

Finally, we obtain the following formulae for the diffusion and the drift terms.

$$\boldsymbol{\Gamma}_{s\pm}(\boldsymbol{r}) \equiv -\mathsf{D}_s\cdot\nabla\omega_\pm \pm \boldsymbol{V}_s\omega_\pm, \tag{60}$$

$$\mathsf{D}_s = \frac{\Omega}{(2\pi)^3}\left(\frac{\pi}{L}\right)^2\frac{1}{k_{\min}}\int d\boldsymbol{r}'\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')(\omega_+'-\omega_-')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}, \tag{61}$$

$$\boldsymbol{V}_s = \frac{\Omega}{(2\pi)^3}\left(\frac{\pi}{L}\right)^2\frac{1}{k_{\min}}\int d\boldsymbol{r}'\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')\cdot\nabla'\omega'}{|\boldsymbol{u}-\boldsymbol{u}'|^3}. \tag{62}$$

It is worth stressing that the term $\boldsymbol{V}_s\omega_+$ has the opposite sign to the term $\boldsymbol{V}_s\omega_-$. It may provide a mechanism for the "charge separation" which is usually seen in the equilibrium distribution for systems with positive and negative vortices [8, 18]. The notations $\boldsymbol{\Gamma}_{s\pm}$ are

defined in the same manner as Eqs. (51). In Eqs. (61) and (62), two unknown parameters L and $k_{\min}$ remain. We assume that $L = gR$ where g is a size factor ($g \ll 1$). If we set $g = 1/4\pi$, total diffusion flux is rewritten as

$$\begin{aligned}\boldsymbol{\Gamma}_s(\boldsymbol{r}) &= \boldsymbol{\Gamma}_{s+} + \boldsymbol{\Gamma}_{s-} \\ &= -\frac{\Omega}{R}\int d\boldsymbol{r}'\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3} \\ &\quad \cdot \left[(\omega'_+ - \omega'_-)\nabla\omega - (\omega_+ - \omega_-)\nabla'\omega'\right]. \end{aligned} \tag{63}$$

A. Diffusion flux in local and global equilibrium states

At first, let us examine if the diffusion fluxes (63) disappears in a local equilibrium state. We rewrite Eq. (63) into a symbolic form:

$$\boldsymbol{\Gamma}_s(\boldsymbol{r}) = -\frac{\Omega}{R}\int d\boldsymbol{r}'\boldsymbol{\gamma}[\omega, \psi; \omega', \psi'] \tag{64}$$

where $\boldsymbol{\gamma}$ is a functional of ω, ψ, ω' and ψ'. Consider a state where temperature is locally uniform in each small region in the system. We call this state the local equilibrium state in which the local equilibrium condition is satisfied:

$$\omega_{\mathrm{leq}\pm} = \omega_{0\pm}\exp(\mp\beta\Omega\psi_{\mathrm{leq}}). \tag{65}$$

Inserting Eqs. (65) into $\boldsymbol{\gamma}$ in Eq. (64), we find that

$$\begin{aligned}&\boldsymbol{\gamma}[\omega_{\mathrm{leq}}, \psi_{\mathrm{leq}}; \omega'_{\mathrm{leq}}, \psi_{\mathrm{leq}}] \\ =& \frac{(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})}{|\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}}|^3} \\ &\times(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})\cdot[(\omega'_{\mathrm{leq}+} - \omega'_{\mathrm{leq}-})\nabla(\omega_{\mathrm{leq}+} + \omega_{\mathrm{leq}-}) - (\omega_{\mathrm{leq}+} - \omega_{\mathrm{leq}-})\nabla'(\omega'_{\mathrm{leq}+} + \omega'_{\mathrm{leq}-})] \\ =& -\beta\Omega(\omega'_{\mathrm{leq}+} - \omega'_{\mathrm{leq}-})(\omega_{\mathrm{leq}+} - \omega_{\mathrm{leq}-}) \\ &\times\frac{(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})}{|\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}}|^3}(\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}})\cdot(\nabla\psi_{\mathrm{leq}} - \nabla'\psi'_{\mathrm{leq}}) \\ =& \, 0 \end{aligned} \tag{66}$$

where $\boldsymbol{u}_{\mathrm{leq}} = -\hat{\boldsymbol{z}}\times\nabla\psi_{\mathrm{leq}}$ is used. As $\boldsymbol{u}_{\mathrm{leq}} - \boldsymbol{u}'_{\mathrm{leq}}$ is perpendicular to $\nabla\psi_{\mathrm{leq}} - \nabla'\psi'_{\mathrm{leq}}$, $\boldsymbol{\gamma}$ is equal to zero and this result indicates that the detailed balance is achieved. When the system reaches a global thermal equilibrium state with uniform β [8]

$$\omega_{\mathrm{eq}\pm} = \omega_{0\pm}\exp(\mp\beta\Omega\psi_{\mathrm{eq}}), \tag{67}$$

we obtain

$$\begin{aligned}\nabla'\omega'_{\rm eq} &= \nabla'(\omega'_{\rm eq+} + \omega'_{\rm eq+}) \\ &= -\beta\Omega(\omega'_{\rm eq+} - \omega'_{\rm eq-})(\nabla'\psi'_{\rm eq} - \nabla\psi_{\rm eq} + \nabla\psi_{\rm eq}). \end{aligned} \tag{68}$$

As $(\boldsymbol{u}_{\rm eq} - \boldsymbol{u}'_{\rm eq}) \cdot (\nabla'\psi'_{\rm eq} - \nabla\psi_{\rm eq}) = 0$, the drift term in Eq. (62) is rewritten as

$$\boldsymbol{V}_{s,\rm eq} = -\beta\Omega\mathsf{D}_{s,\rm eq} \cdot \nabla\psi_{\rm eq} \tag{69}$$

which is a counterpart of the Einstein relation [10].

B. Energy-conservative property of collision term

Time derivative of the total mean field energy E is given by

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'G(\boldsymbol{r}-\boldsymbol{r}')\left(\frac{\partial\omega'}{\partial t}\omega + \omega'\frac{\partial\omega}{\partial t}\right) \\ &= \int d\boldsymbol{r}\psi\frac{\partial\omega}{\partial t}, \end{aligned} \tag{70}$$

where

$$\begin{aligned} E &\equiv \frac{1}{2}\int d\boldsymbol{r}\psi\omega \\ &= \frac{1}{2}\int d\boldsymbol{r}\int d\boldsymbol{r}'G(\boldsymbol{r}-\boldsymbol{r}')\omega'\omega. \end{aligned} \tag{71}$$

Inserting the space-averaged equation of motion

$$\frac{\partial\omega}{\partial t} + \nabla\cdot(\boldsymbol{u}\omega) = -\nabla\cdot\boldsymbol{\Gamma}_s \tag{72}$$

into Eq. (70), we obtain

$$\begin{aligned}\frac{dE}{dt} &= \int d\boldsymbol{r}\psi\left(-\nabla\cdot(\boldsymbol{u}\omega) - \nabla\cdot\boldsymbol{\Gamma}_s\right) \\ &= \int d\boldsymbol{r}\nabla\psi\cdot\boldsymbol{u}\omega + \int d\boldsymbol{r}\nabla\psi\cdot\boldsymbol{\Gamma}_s \\ &= \int d\boldsymbol{r}\nabla\psi\cdot\boldsymbol{\Gamma}_s \\ &= -\frac{\Omega}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\nabla\psi\cdot\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\cdot\left[(\omega'_+ - \omega'_-)\nabla\omega - (\omega_+ - \omega_-)\nabla'\omega'\right]. \end{aligned} \tag{73}$$

By permuting the dummy variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ in Eq. (73) and taking the half-sum of the resulting expressions, we obtain

$$\begin{aligned}\frac{dE}{dt} &= -\frac{\Omega}{2R}\int d\boldsymbol{r}\int d\boldsymbol{r}'(\nabla\psi - \nabla'\psi')\cdot\frac{\boldsymbol{u}-\boldsymbol{u}'}{|\boldsymbol{u}-\boldsymbol{u}'|^3} \\ &\quad\times(\boldsymbol{u}-\boldsymbol{u}')\cdot\left[(\omega'_+ - \omega'_-)\nabla\omega - (\omega_+ - \omega_-)\nabla'\omega'\right] \\ &= 0. \end{aligned} \tag{74}$$

We conclude that the obtained diffusion fluxes conserve the total mean field energy.

C. H theorem

The entropy function S is defined by using the H function:

$$S = -k_B H, \tag{75}$$

$$\begin{aligned} H &= \int d\boldsymbol{r} \frac{\omega_+}{\Omega} \ln \frac{\omega_+}{\Omega} + \frac{\omega_-}{-\Omega} \ln \frac{\omega_-}{-\Omega} + \text{const.} \\ &= \frac{1}{\Omega} \int d\boldsymbol{r} \omega_+ \ln \omega_+ - \omega_- \ln |\omega_-| - 2N \ln \Omega + \text{const.} \end{aligned} \tag{76}$$

The time derivative of the H function is given by

$$\begin{aligned} \frac{dH}{dt} &= \frac{1}{\Omega} \int d\boldsymbol{r} \frac{\partial \omega_+}{\partial t} (\ln \omega_+ + 1) - \frac{\partial \omega_-}{\partial t} (\ln |\omega_-| + 1) \\ &= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{u} \omega_+ \cdot \nabla \ln \omega_+ - \boldsymbol{u} \omega_- \cdot \nabla \ln |\omega_-| \\ &\quad + \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_+ - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_-| \\ &= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{u} \cdot \nabla \omega_+ - \boldsymbol{u} \cdot \nabla \omega_- \\ &\quad + \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_+ - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_-| \\ &= \frac{1}{\Omega} \int d\boldsymbol{r} \boldsymbol{\Gamma}_{s+} \cdot \nabla \ln \omega_+ - \boldsymbol{\Gamma}_{s-} \cdot \nabla \ln |\omega_-|. \end{aligned} \tag{77}$$

For simplicity, we introduce the following notations.

$$\begin{aligned} &\nabla \ln \omega_+ = \boldsymbol{v}_+, \quad \nabla \ln |\omega_-| = \boldsymbol{v}_-, \\ &\boldsymbol{u} - \boldsymbol{u}' = \boldsymbol{U}. \end{aligned} \tag{78}$$

Inserting Eq. (63) into Eq. (77), we obtain

$$\begin{aligned} \frac{dH}{dt} &= -\frac{1}{R} \int d\boldsymbol{r} \int d\boldsymbol{r}' \boldsymbol{v}_+ \cdot \frac{\boldsymbol{U}\boldsymbol{U}}{|\boldsymbol{U}|^3} \cdot [(\omega'_+ - \omega'_-) \nabla \omega_+ - \omega_+ \nabla' \omega'] \\ &\quad - \frac{1}{R} \int d\boldsymbol{r} \int d\boldsymbol{r}' \boldsymbol{v}_- \cdot \frac{\boldsymbol{U}\boldsymbol{U}}{|\boldsymbol{U}|^3} \cdot [(\omega'_+ - \omega'_-) \nabla \omega_- + \omega_- \nabla' \omega'] \\ &= -\frac{1}{R} \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{1}{|\boldsymbol{U}^3|} \\ &\quad \times \Big[\omega_+ \omega'_+ (\boldsymbol{v}_+ \cdot \boldsymbol{U}\boldsymbol{U} \cdot (\boldsymbol{v}_+ - \boldsymbol{v}'_+) \\ &\quad + \omega_- \omega'_- (\boldsymbol{v}_- \cdot \boldsymbol{U}\boldsymbol{U} \cdot (\boldsymbol{v}_- - \boldsymbol{v}'_-) \\ &\quad - \omega_+ \omega'_- (\boldsymbol{v}_+ \cdot \boldsymbol{U}\boldsymbol{U} \cdot (\boldsymbol{v}_+ + \boldsymbol{v}'_-) \\ &\quad - \omega_- \omega'_+ (\boldsymbol{v}_- \cdot \boldsymbol{U}\boldsymbol{U} \cdot (\boldsymbol{v}_- + \boldsymbol{v}'_+) \Big]. \end{aligned} \tag{79}$$

By permuting the dummy variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ in Eq. (79) and taking the half-sum of the resulting expressions, we obtain

$$\frac{dH}{dt} = -\frac{1}{2R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{1}{|\boldsymbol{U}^3|} \times[\omega_+\omega'_+|(\boldsymbol{v}_+ - \boldsymbol{v}'_+)\cdot\boldsymbol{U}|^2 + \omega_-\omega'_-|(\boldsymbol{v}_- - \boldsymbol{v}'_-)\cdot\boldsymbol{U}|^2 - \omega_+\omega'_-|(\boldsymbol{v}_+ - \boldsymbol{v}'_-)\cdot\boldsymbol{U}|^2 - \omega_-\omega'_+|(\boldsymbol{v}_- - \boldsymbol{v}'_+)\cdot\boldsymbol{U}|^2] \leq 0. \tag{80}$$

The integrand of Eq. (80) is positive or equal to zero, and dH/dt is negative or equal to zero. It is concluded that the entropy function (75) is the monotonically increasing function.

VI. DISCUSSION

We have demonstrated the simple and explicit formula of the Fokker-Planck type collision term for nonaxisymmetric, double-species point vortex system without the collective effect.

Again, it should be noted that the drift term plays an important role in the inverse cascade process in 2D turbulence. In Eq. (60), the sign of the drift term changes following the sign of the vorticity, while the sign of the diffusion term is always negative. This implies that the drift term provides the “charge separation” of the vortices, which is commonly observed in equilibrium states at negative temperature.

If the sign of β is negative, $d\omega_{\rm eq}/d\psi_{\rm eq} \geq 0$, as

$$\frac{d\omega_{\rm eq}}{d\psi_{\rm eq}} = -\beta\Omega(\omega_{\rm eq+} - \omega_{\rm eq-}) \tag{81}$$

where

$$\omega_{\rm eq} = \omega_{\rm eq+} + \omega_{\rm eq-} \tag{82}$$

and the relations (67) are used. The point vortex system easily approaches a quasi-stationary state near the thermal equilibrium state by a violent relaxation, which is purely collisionless and driven by mean field effects. The state is characterized by

$$\nabla\cdot(\boldsymbol{u}\omega) = 0 \tag{83}$$

or equivalently

$$\omega = \omega(\psi). \tag{84}$$

In this state, we may expect that

$$\frac{d\omega}{d\psi} \geq 0 \tag{85}$$

almost everywhere in the system. If the above postulate is justified, the following two conclusions are drawn which elucidate the role of the drift term at negative temperature.

Energy conservation is achieved by the energy dissipation process due to the diffusion term and the energy production process due to the drift term. We divide the expression (73) into two parts, namely the term which corresponds to the diffusion term and the one to the drift term.

$$\frac{dE}{dt} = \left.\frac{dE}{dt}\right|_{\mathsf{D}} + \left.\frac{dE}{dt}\right|_{\boldsymbol{V}} = 0 \tag{86}$$

$$\left.\frac{dE}{dt}\right|_{\mathsf{D}} = -\frac{\Omega}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\nabla\psi\cdot\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\cdot(\omega'_+ - \omega'_-)\nabla\omega \tag{87}$$

$$\left.\frac{dE}{dt}\right|_{\boldsymbol{V}} = \frac{\Omega}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\nabla\psi\cdot\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\cdot(\omega_+ - \omega_-)\nabla'\omega' \tag{88}$$

If the vorticity ω is a function of the stream function ψ (see Eq. (84)), Eqs. (87) and (88) are rewritten as:

$$\left.\frac{dE}{dt}\right|_{\mathsf{D}} = -\frac{\Omega}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{|\nabla\psi\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{|\boldsymbol{u}-\boldsymbol{u}'|^3}(\omega'_+ - \omega'_-)\frac{d\omega}{d\psi} \tag{89}$$

$$\left.\frac{dE}{dt}\right|_{\boldsymbol{V}} = \frac{\Omega}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{|\nabla\psi\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{|\boldsymbol{u}-\boldsymbol{u}'|^3}(\omega_+ - \omega_-)\frac{d\omega'}{d\psi'} \tag{90}$$

Thus, if $d\omega/d\psi \geq 0$, it is concluded that

$$\left.\frac{dE}{dt}\right|_{\boldsymbol{V}} = -\left.\frac{dE}{dt}\right|_{\mathsf{D}} \geq 0 \tag{91}$$

Similarly, the diffusion term increases the entropy, while the drift term decreases it. We divide the expression (77) into two parts (the entropy function S is used instead of the H function).

$$\frac{dS}{dt} = \left.\frac{dS}{dt}\right|_{\mathsf{D}} + \left.\frac{dS}{dt}\right|_{\boldsymbol{V}} \geq 0 \tag{92}$$

$$\begin{aligned}\left.\frac{dS}{dt}\right|_{\mathsf{D}} &= \frac{k_B}{R}\int d\boldsymbol{r}\frac{\nabla\omega_+\cdot\mathsf{D}\cdot\nabla\omega_+}{\omega_+} - \frac{\nabla\omega_-\cdot\mathsf{D}\cdot\nabla\omega_-}{\omega_-} \\ &= \frac{k_B}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{|\nabla\omega_+\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{\omega_+} - \frac{|\nabla\omega_-\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{\omega_-}\end{aligned} \tag{93}$$

$$\begin{aligned}\left.\frac{dS}{dt}\right|_{\boldsymbol{V}} &= -\frac{k_B}{R}\int d\boldsymbol{r}\boldsymbol{V}\cdot\nabla\omega \\ &= -\frac{k_B}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\nabla\omega\cdot\frac{(\boldsymbol{u}-\boldsymbol{u}')(\boldsymbol{u}-\boldsymbol{u}')}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\cdot\nabla'\omega'\end{aligned} \tag{94}$$

Equation (93) indicates that

$$\left.\frac{dS}{dt}\right|_{\mathsf{D}} \geq 0 \tag{95}$$

regardless of the sign of $d\omega/d\psi$. On the other hand, if the assumptions (84) and (85) are valid, Eq. (94) is rewritten as

$$\left.\frac{dS}{dt}\right|_{\boldsymbol{V}} = -\frac{k_B}{R}\int d\boldsymbol{r}\int d\boldsymbol{r}'\frac{|\nabla\psi\cdot(\boldsymbol{u}-\boldsymbol{u}')|^2}{|\boldsymbol{u}-\boldsymbol{u}'|^3}\frac{d\omega}{d\psi}\frac{d\omega'}{d\psi'} \tag{96}$$

and we obtain

$$\left.\frac{dS}{dt}\right|_{\boldsymbol{V}} \leq 0. \tag{97}$$

It is also concluded that in the thermal equilibrium state with $\beta < 0$,

$$\left.\frac{dS}{dt}\right|_{\boldsymbol{V}} = -\left.\frac{dS}{dt}\right|_{\mathsf{D}} \leq 0. \tag{98}$$

This may indicates that to form vortex clumps, a background distribution outside the clumps is necessary to dump the entropy. This conclusion is supported by the nonneutral plasma experiments [19–22] and the numerical simulation [18].

There are several outstanding issues remaining. First, the final formulae (61) and (62) include unknown parameters k_{min} and L. Second, the integrals in Eqs. (61) for D and (62) for $\boldsymbol{V}$ contain the divergent integrand, although combined terms $\boldsymbol{\Gamma}_{s\pm} = -\mathsf{D}_s\cdot\nabla\omega_\pm \pm \boldsymbol{V}_s\omega_\pm$ are regularized. A more rigorous justification will be needed for fixing the above two issues.

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